

**MULTI SOLITONS SOLUTIONS OF  
KORTEWEWG de VRIES (KdV) EQUATION: SIX SOLITONS**

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*To my beloved husband, Zahari bin Samsudin, my future child and  
parents.*

*Thank you for moral supporting me all the way and may Allah bless you.*

*To my supervisor, Assoc. Prof. Dr. OngCheeTiong.*

*Thank you for guiding me in my thesis.*

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## ABSTRACT

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation has nonlinearity and dispersion effects. The balance between these effects leads to a wave propagation that is soliton solution. It propagates without changing its shape. The purpose of this research is to obtain the multi solitons solutions of KdV equation up to six-solitons solutions. The Hirota's bilinear method will be implemented to find the explicit expression for up to six-solitons solutions of KdV equation. Identification of the phase shift that makes full interactions happens at  $x = 0$  and  $t = 0$  for each multi soliton solution of KdV equation. The Maple computer programming will be used to produce the various interactive graphical outputs for up to six-solitons solutions of KdV equation.

## ABSTRAK

Persamaan Korteweg de-Vries (KdV) adalah persamaan terbitan separa tak linear yang mempunyai kesan tak linear dan penyelerakan. Keseimbangan antara kesan ini membawa kepada perambatan gelombang iaitu penyelesaian soliton. Gelombang merambat tanpa berubah bentuknya. Tujuan kajian ini adalah untuk mendapatkan penyelesaian multi soliton bagi persamaan KdV sehingga penyelesaian enam soliton. Kaedah Hirota bilinear akan diguna pakai untuk mencari penyelesaian eksplisit bagi penyelesaian sehingga enam-soliton dalam persamaan KdV. Identifikasi anjakan fasa ketika interaksi penuh berlaku pada  $x = 0$  and  $t = 0$  bagi setiap penyelesaian multi soliton persamaan KdV. Pengaturcaraan komputer, MAPLE akan digunakan untuk menghasilkan pelbagai paparan grafik yang interaktif bagi penyelesaian sehingga enam-soliton dalam persamaan KdV.

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## LISTS OF ABBREVIATIONS/ SYMBOLS

$A_k$	-	Antikink.
DNA	-	Deoxyribo Nucleic Acid.
IST	-	Inverse Scattering Transform.
fKdV	-	Forced Korteweg de-Vries.
HAM	-	Homotopy Analysis Method.
HPM	-	Homotopy Perturbation Method.
KdV	-	Korteweg de-Vries.
KP	-	Kadomtsev Petviashvili.
K	-	Kink.
nPDE	-	Nonlinear Partial Differential Equation.
PDE	-	Partial Differential Equation.
VIM	-	Variational Iteration Method.
2D	-	Two dimensional.
3D	-	Three dimensional.
$u(x,t)$	-	The elongation of the wave function with variable $x$ and $t$ .
$x$	-	Space domain.
$t$	-	Time domain.
$\eta$	-	The height of the peak.
$c$	-	Speed of the wave.
$\delta$	-	Phase shift of the soliton.
$l$	-	The depth of the water.
$g$	-	The gravitational acceleration.
$k$	-	Amplitude of soliton.



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## **CHAPTER 1**

### **INTRODUCTION**

This chapter gives an explanation about introduction of solitary wave. Besides, this chapter discusses the background and the problem statement of study. Next, it will explain the objective of study and the scope of study.

#### **1.1 Preface**

In 1834, a famous hydrodynamic John Scott Russell was described the solitary wave from his first observation while he was riding his horse along a canal near Edinburgh. He had renamed this solitary wave phenomenon and called it “the wave of translation”. He concluded the solitary wave speed depends on the amplitude or height of wave.

On the other hand, he contends that his discovery about solitary wave was a real revelation. But, at that time his enthusiasm not interest by many people to explore the solitary wave phenomenon more detail. About more than 100 years, the discovery of solitary wave was beheld by mathematician and physician to realize the importance of John Scott Russell discovery.

In year 1895, John Scott Russell's theoretical understanding has been studied by Diederik Johannes Korteweg and his PhD student, Gustav de Vries. They have derived an equation that related with John Scott Russell's theoretical understanding. This equation named same as they name which is Korteweg de-Vries equation. It also called as the KdV equation. The feature of the KdV equation is the speed of solitary wave proportional to their amplitude. Thus, the higher amplitude of solitary wave will move faster than the shorter solitary wave (Daouxious &Peyrard, 2006).

In 1965, the famous America physicist, Norman Zabusky and the physicist, Martin Kruskal were published the numerical solution that discovered the solitary waves to maintain the shape after the interaction occurred. They also invented the term 'soliton' because the solitary waves have unchangeable property as the collision of particles. In generally, this term has been accepted and correctly revealed the substance of the solitary waves.

The mathematician and physician have given a lot of effort in the solitary wave field such as fluid dynamics, elementary particles physics, plasma physics and others. Thus, these efforts generate a few of nonlinear evolution equation such as Korteweg de-Vries (KdV) equation, Sine-Gordon equation, KadomtsevPetviashvili (KP) equation and other equation that have been developed from the KdV equation (Chaohao, 1995).

## **1.2 Background of Study**

Since 1965, Norman Zabusky and Kruskal discovered the important behaviour of soliton by computation. The numerical studies of nonlinear waves have developed widely and appear as one of the active branches of numerical analysis. There exist three main numerical methods are finite difference method, finite element method and spectral method.

As mentioned above, we found many researchers had done various investigations on the KdV equation. The KdV equation can model the dynamics of solitary waves. This equation is a nonlinear, dispersive and non-dissipative equation which has soliton solutions. The standard form of the KdV equation can be written as

$$u_t + \alpha uu_x + \beta u_{xxx} = 0 \quad (1.0)$$

where  $u(x, t)$  refers to the elongation of the wave at place,  $x$  and at time,  $t$ . The term  $u_t$  describes the move frame of the wave. Besides,  $\alpha uu_x$  is the nonlinear term which the wave propagates with a speed proportional to  $\alpha u$ . The  $\beta u_{xxx}$  term generates dispersive broadening that can exactly compensate the narrowing caused by nonlinear term under the proper condition. The variables  $\alpha$  and  $\beta$  are parameter or constant coefficient. (Kolebaje & Oyewande, 2012).

Thus, in this research will investigate the analytical solution of KdV equation by using Hirota's bilinear method for multi-solitons. The KdV equation that will be used in this research is

$$u_t + uu_x + 6u_{xxx} = 0$$

where  $\beta$  equal six. Six is the constant coefficient.

In years 1971, a Japanese researcher, Hirota developed a new direct method for constructing the multi-solitons solution of KdV equation. He also derived an explicit expression of multi-solitons solutions. This new direct method is called the Hirota's bilinear method. The Hirota's method is effective and fastest way to produce the results of KdV equation solution for multi-solitons (Matsuno, 1984).

### 1.3 Statement of Problem

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation that can be solved numerically and analytically. To obtain the solutions of the KdV equation is not easy. In this research, we need to observe the soliton ladder of solutions. Then, to get the two, three, four, five and six-solitons solutions by using Hirota's bilinear method, here we find to produce the permutation parameters of solitons. The two, three, four, five and six-solitons solutions are difficult and complicated to calculate manually. So, we need to use the computer programming tools to derive the  $f$  function and produce the various interactive graphical outputs for up to six-solitons solutions of KdV equation.

### 1.4 Objectives of Study

In this research, we focus on three main objectives of the research. There are

- a) Solving the KdV equation by using the Hirota's bilinear method.
- b) Obtaining the multi-solitons solutions and the graphical outputs of KdV equation for up to six solitons.
- c) Analysing the conservation laws of mass, momentum and energy for one soliton solution of KdV equation.

### 1.5 Scope of Study

In this research, we consider the KdV equation which is written as below

$$u_t + uu_x + 6u_{xxx} = 0.$$

We wish to investigate the solutions of KdV equation up to six-solitons by using the Hirota's bilinear method. The various graphical outputs of up to six-solitons solutions will be studied.

## **1.6 Significance of Study**

Mainly, this research will discuss more about the multi-solitons solutions of KdV equation up to six-solitons. The Hirota's bilinear method will be used to obtain these solutions of KdV equation. Soliton or also known as solitary wave growth in broad field such as shallow and deep water waves, fibre optics, protein and DNA, magnet, bions, and biological models.

The KdV equation is a nonlinear, dispersion and non-dissipation equation which has the soliton solutions. The nonlinear effect and dispersion effect gives important roles in various fields such as tsunamis phenomenon. The balancing between nonlinearity and dispersion effects in the KdV equation important to make the waves maintain their shape after a collision occurs.

For instance, the balance between the effects of nonlinearity and dispersion show why the tsunamis spread out their waves after travelling a long distance along the beach with different depth of sea. The travelling of tsunamis waves behaving as a solitons and its can be modelled as a KdV equation. Then, this research problem will help us to solve certain problem in tsunamis phenomenon.

Through this research, we will able to obtain solitons ladders for up to six-solitons of KdV equation by using Hirota bilinear method. Besides, we will able to verify the conservation laws of mass, momentum and energy for multi-solitons solutions

of KdV equation. In addition, we also able to see symmetrical patterns in soliton solutions due the permutation of  $f$  function.

## REFERENCES

- Abdou, M. A. (2011). *Multi Kink Solutions and Multiple Singular Kink Solutions for (2+1)- dimensional Integrable Breaking Soliton Equations by Hirota's Method*. Journal of Nonlinear Sciences. 2(1): 1-4.
- Ballou, G. (2005). *Handbook for Sound Engineer (3 edition)*. Focal Press-Gulf Professional Publishing: 1499.
- Borhanifar, A. and Kabir, M. M. (2009). *New Periodic and Soliton by Application of Exp-function Method for Nonlinear Evolution Equation*. Journal of Computational and Applied Mathematics. 229: 158-167.
- Boling, G. (1995). *Soliton Theory and Modern Physics*. Springer-Zhejiang Science & Technology Publishing House: 1-5.
- Brauer, K. (2000). *The Korteweg-de Vries Equation: History, Exact Solutions, and Graphical Representation*. Germany: University of Osnabruck.
- Chaohao, G. (1995). *Soliton Theory and Its Applications*. Springer-Zhejiang Science & Technology Publishing House: 1-4.
- Dauxois, T. and Peyrard, M. (2006). *Pyhsics of Solitons*. United Kingdom: Cambridge University Press: 1-17.



- Enns, R. H. and McGuire, G. C. (2001). *Nonlinear Physics with Mathematica for Scientists and Engineers: Nonlinear Systems. Part II*. Springer-Birkhäuser: 96-102.
- Fan, E. G. (2000). *Two New Applications of the Applications of the Homogenous Balance Method*, Phys. Lett. A. 265: 353-357.
- Hereman, W. and Nuseir, A. (1997). *Symbolic Methods to construct exact solutions of Nonlinear Partial Differential Equation*. Journal of Computational Applied Mathematics. 43: 13-47.
- Hirota, R. (1980). *Direct Method in Soliton Theory, in Soliton*. New York: Springer.
- Holden, H., Karlsen, K. H. and Risebro, N. H. (1999). *Operator Splitting Method for Generalized Korteweg-de Vries Equation*. Journal of Computational Physics. 153: 203-222.
- Inan, I. E. and Kaya, D. (2006). *Some Exact Solutions to the Potential Kadomtsev Petviashvili Equation and to a System of Shallow Water Wave Equation*. Phys. Lett. A. 320: 314-318.
- Jawad, M. A. J., Petkovic, M. D. and Biswas, A. (2011). *Soliton Solutions for Nonlinear Calogero-Degasperis and Potential Kadomtsev-Petviashvili Equation*. Journal of Computational Mathematical Applied: 62: 2621-2628.
- Kaya, D. and El-Sayed, S. M. (2003). *Numerical soliton-like solutions of the Potential Kadomtsev-Petviashvili Equation by the decomposition Method*. Phys. Lett. A. 320: 192-199.

- Kolebaje, O. and Oyewande, O. (2012). *Numerical Solution of The Korteweg De Vries Equation by Finite Difference and Adomian Decomposition Method*. International Journal of Basic and Applied Sciences: 321-335.
- Konopelchenko, G. (1993). *Soliton in Multi-dimensional*. Singapore: World Scientific.
- Lomdahl, P. S. (1984). *What is Soliton?*. Journal of Los Alamos Science: 27-31.
- Marchant, T. R. (1999). *Asymptotic Solitons of the Extended Korteweg-de Vries Equation*. University of Wollongong: Journal of the Mathematics and Applied Statistics. 59(3): 3745.
- Matsumo, Y. (1984). *Bilinear Transformation Method*. London: Academic Press Inc.: 1-46.
- Miles, J. W. (1981). *The Korteweg-de Vries: A Historical Essay*. Journal of Fluid Dynamic. 106: 131-147.
- Miroshnichenko, A. E., Vasiliev, A. A. and Dmitriev, S. V. *Soliton and Soliton Collisions*. Russian. Retrieved from [homepages.tversu.ru/~s000154/collision/main.html](http://homepages.tversu.ru/~s000154/collision/main.html).
- Miura, R. M. (1978). *An Introduction to Solitons and the Inverse Scattering Method via the Korteweg-de Vries Equation*. Academic Press, Inc.: 1-5.
- Mohamed, Z. (2002). *Persamaan Korteweg-de Vries: Satu Tinjauan*. Universiti Teknologi Malaysia: M.Sci. Thesis.

Najafi, M. (2012). *Multiple Soliton Solutions of Second-Order Benjamin-Ono Equation*.  
Journal of Applied Engineering Mathematics. 2(1): 94-100.

Newell, A. C. (1985). *Solitons in Mathematics and Physics*. University of Arizona: 120-122.

Ong, C. T. (2002). *Development of Numerical Package (FORSO) and Its Applications on Forced Korteweg-de Vries and Other Nonlinear Evolution Equation*.  
Universiti Teknologi Malaysia: Phd. Thesis.

Ong, C.T. (1993). *Various Aspects of Solitons Interactions*. Universiti Teknologi Malaysia: M.Sci. Thesis.

Saadi, F., Azizpour, M. J. and Zahedi, S. A. (2010). *Analytical Solutions of Korteweg-de Vries (KdV) Equation*. World Academy of Science-Engineering and Technology 45: 171-175.

Wazwaz, A. M. (2008). *Multiple-Soliton Solutions for the Ninth-Order KdV Equation and Sixth-Order Boussineq*. Journal of the Applied Mathematics and Computational. 204: 20-26.

Wazwaz, A.M. (2008). *Single and Multiple-Soliton Solutions for the (2+1)-dimensional KdV Equation*. Journal of the Applied Mathematics and Computational. 203(1): 277-283.

Zhang, J. (2012). *Using the Simplified Hirota's Method to Investigate Multi-Solitons of the Fifth-order KdV equation*. International Mathematical Forum. 7: 917-924.

Zhen, J. Z., Jing, Z. F. and Zhi, B. L. (2010). *Maple Soliton Solution for Computing Hirota's Bilinear Equation and Multisoliton Solution of Nonlinear Evolution Equations*. Journal of the Applied Mathematics and Computational. 217(1): 92-104.